3 January 2002
PHYSICS LETTERS B
ELSEVIER
Physics Letters B 524 (2002) 208–216
www.elsevier.com/locate/npe

Complete $\mathcal{N}=4$ structure of low-energy effective action in $\mathcal{N}=4$ super-Yang–Mills theories

I.L. Buchbinder [a], E.A. Ivanov [b]

[a] *Department of Theoretical Physics, Tomsk State Pedagogical University, Tomsk 634041, Russia*
[b] *Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, 141980 Moscow Region, Russia*

Received 16 November 2001; accepted 22 November 2001
Editor: P.V. Landshoff

Abstract

Using the $\mathcal{N}=2$ superfield approach, we construct full $\mathcal{N}=4$ supersymmetric low-energy effective actions for $\mathcal{N}=4$ SYM models, with both $\mathcal{N}=2$ gauge superfield strengths and hypermultiplet superfields included. The basic idea is to complete the known non-holomorphic effective potentials which depend only on $\mathcal{N}=2$ superfield strengths W and $\overline{W}$ to the full on-shell $\mathcal{N}=4$ invariants by adding the appropriate superfield hypermultiplet terms. We prove that the effective potentials of the form $\ln W \ln \overline{W}$ can be $\mathcal{N}=4$ completed in this way and present the precise structure of the corresponding completions. However, the effective potentials of the non-logarithmic form suggested in hep-th/9811017 and hep-th/9909020 do not admit the $\mathcal{N}=4$ completion. Therefore, such potentials cannot come out as (perturbative or non-perturbative) quantum corrections in $\mathcal{N}=4$ SYM models.

1. Extended rigid supersymmetry imposes very strong restrictions on the structure of quantum corrections in the corresponding field theories. It is natural to expect that the strongest restrictions occur in a theory possessing the maximally extended rigid supersymmetry, i.e., in $\mathcal{N}=4$ super Yang–Mills theory. In principle, these restrictions could be so powerful that allow one to find the exact solutions for some physical objects of interest, like low-energy effective action or correlation functions, based solely on the supersymmetry reasonings.

The study of low-energy effective action of $\mathcal{N}=4$ SYM models was initiated in [1].[1] In the $\mathcal{N}=2$ formulation, the full $\mathcal{N}=4$ gauge multiplet consists of the $\mathcal{N}=2$ gauge multiplet and hypermultiplet. The authors of [1] studied the effective action of $\mathcal{N}=4$ SYM theory with the gauge group $SU(2)$ spontaneously broken to $U(1)$ and considered that part of this action which depends only on the fields of massless $U(1)$ $\mathcal{N}=2$ vector multiplet. The requirements of scale and R-invariances determine this part of the effective action up to a numerical coefficient. The result can be given in terms of non-holomorphic effective potential

$$\mathcal{H}(W,\overline{W}) = c \ln\frac{W}{\Lambda}\ln\frac{\overline{W}}{\Lambda}, \tag{1}$$

where W and $\overline{W}$ are the $\mathcal{N}=2$ superfield strengths, Λ is an arbitrary scale and c is an arbitrary real constant. The effective action defined as an integral of $\mathcal{H}(W,\overline{W})$ over the full $\mathcal{N}=2$ superspace with the

E-mail addresses: joseph@tspu.edu.ru (I.L. Buchbinder), eivanov@thsun1.jinr.ru (E.A. Ivanov).

[1] By the low-energy effective action we always mean the leading in the external momenta piece of the full quantum effective action.

0370-2693/02
PII: S0370-2693(01)01388-0

coordinates $z = (x^m, \theta_{\alpha i}, \bar{\theta}^i_{\dot{\alpha}})$ is independent of the scale Λ. It is worth pointing out that the result (1) was obtained in $N = 4$ SYM theory entirely on the symmetry grounds. [2]

Eq. (1) provides the *exact* form of the low-energy effective action. Any quantum corrections must be absorbed into the coefficient c. One can show [1,9] that the non-holomorphic effective potential (1) gets neither perturbative nor non-perturbative contributions beyond one loop. As the result, construction of exact low-energy effective action for the $SU(2)$ SYM theory in the Coulomb branch (i.e., with $SU(2)$ broken to $U(1)$) is reduced to computing the coefficient c in the one-loop approximation.

The direct derivation of the potential (1), computation of the coefficient c and, hence, the final reconstruction of the full exact low-energy $U(1)$ effective action from the quantum $\mathcal{N} = 4$ SYM theory were undertaken in Refs. [3–5]. Further studies showed that the result (1), obtained for the gauge group $SU(2)$ spontaneously broken to $U(1)$, can be generalized to the group $SU(N)$ broken to its maximal abelian subgroup [6–9]. The corresponding one-loop effective potential is given by

$$\mathcal{H}(W, \overline{W}) = c \sum_{I<J} \ln \frac{W^I - W^J}{\Lambda} \ln \frac{\overline{W}^I - \overline{W}^J}{\Lambda}, \qquad (2)$$

with the same coefficient c as in (1) for $SU(2)$ group. Here $I, J = 1, 2, \ldots, N$, $W = \sum_I W^I e_{II}$ belongs to Cartan subalgebra of the algebra $su(N)$, $\sum_I W^I = 0$, and e_{IJ} is the Weyl basis in $su(N)$ algebra (for details see Ref. [8]).

Although the potential (2) looks quite analogous to (1), we cannot state that (2) determines the exact low-energy effective action. The same arguments [1,9] which suggest the absence of higher-loop corrections to c in Eq. (1) equally apply to the effective potential (2), which thus should be fully specified by one-loop contributions. However, the general scale and R-invariance considerations do not forbid the presence of some extra terms in the non-holomorphic effective potential, those of the form [9,10]

$$f\left(\frac{W^I - W^J}{W^K - W^L}, \frac{\overline{W}^I - \overline{W}^J}{\overline{W}^K - \overline{W}^L}\right), \qquad (3)$$

with f being real functions. Such terms are absent for $SU(2)$ group broken to $U(1)$ because of only one W involved, but they are allowed for any $SU(N)$ group broken to $U(1)^{N-1}$ for $N > 2$ beyond one loop. The direct calculation undertaken in Ref. [11] has not confirmed the appearance of terms like (3) at two, three and four loops. However, in a general setting, the question about a possible contribution of terms (3) to the low-energy effective action remained open. On the other hand, it would be extremely useful to know the full structure of the low-energy effective action of $\mathcal{N} = 4$ SYM theory, e.g., for understanding the form of exact quantum corrections in the hypermultiplet sector and getting a deeper insight into the supergravity/$\mathcal{N} = 4$ SYM correspondence (see [24] and references therein).

We wish to pay attention to the fact that all the results concerning the structure of low-energy effective action of the $\mathcal{N} = 4$ SYM theory were actually obtained for a particular part of it, viz. that containing only $\mathcal{N} = 2$ superfield strengths. Indeed, Eqs. (1)–(3) determine a dependence of effective action only on the fields of abelian $\mathcal{N} = 2$ vector multiplet, dependence on the hypermultiplet fields completing the $\mathcal{N} = 2$ vector multiplets to the $\mathcal{N} = 4$ ones remains undefined. Moreover, the general reasoning adduced in Ref. [1] to fix the form of the effective potential (1) and in Refs. [9,10] to reveal the possibility of extra contributions (3) is equally applicable to any $\mathcal{N} = 2$, $D = 4$ superconformal model, not just to $\mathcal{N} = 4$ SYM theory. The latter, from the standpoint of $\mathcal{N} = 2$ supersymmetry, is a theory of $\mathcal{N} = 2$ vector multiplet minimally coupled to the hypermultiplet in the adjoint representation. Nevertheless, the effective action of $\mathcal{N} = 4$ SYM theory, even in the $\mathcal{N} = 2$ vector multiplet sector, could have a much more restricted form compared to effective actions of other $\mathcal{N} = 2$ models just due to the sever bounds imposed by $\mathcal{N} = 4$ supersymmetry. [3] Not every function of W, $\overline{W}$ admissible within the $\mathcal{N} = 2$ supersymmetry framework, could happen to permit an extension to a full $\mathcal{N} = 4$ invariant. In particular, the contributions of the

[2] Non-holomorphic potentials of the form (1) as possible contributions to the effective action in $N = 2$ SYM theories were earlier considered in Ref. [2].

[3] The fact that such a situation actually takes place was demonstrated in Ref. [19] for some terms in one-loop effective action.

form (3), although being certainly allowed in a generic $\mathcal{N}=2$ superconformal theory, could be ruled out in $\mathcal{N}=4$ SYM theory just for this reason.

The aim of the present Letter is to unveil the full $\mathcal{N}=4$ structure of low-energy effective action in $\mathcal{N}=4$ SYM models and to prove, on this basis, the above conjecture. In view of lacking manifestly $\mathcal{N}=4$ supersymmetric off-shell formulation of $\mathcal{N}=4$ SYM, the natural framework for solving this problem is provided by a formulation of $\mathcal{N}=4$ SYM theory in terms of superfields carrying the off-shell representations of $\mathcal{N}=2$ supersymmetry. These superfields are defined on $\mathcal{N}=2$ harmonic superspace [12–14] which is the only one where all basic $\mathcal{N}=2$ supersymmetric models have off-shell formulations. The harmonic superspace approach was used to study the effective action of $\mathcal{N}=2,4$ supersymmetric gauge theories in Refs. [5,8,11,15–18,20,21].

To find the full $\mathcal{N}=4$ structure of low-energy effective action, we proceed in the following way. We start from the $\mathcal{N}=4$ SYM theory formulated in terms of $\mathcal{N}=2$ harmonic superfields comprising $\mathcal{N}=2$ vector multiplet and hypermultiplet. Such a formulation possesses the manifest off-shell $\mathcal{N}=2$ supersymmetry and also an extra hidden $\mathcal{N}=2$ supersymmetry. They close on on-shell $\mathcal{N}=4$ supersymmetry. Then we examine which terms with the hypermultiplet superfields must be added to the potentials (1)–(3) in order to make the full effective actions $\mathcal{N}=4$ supersymmetric. We show that such extra terms indeed exist for the potentials (1), (2) and find their exact form. At the same time, for the potentials (3) analogous terms cannot be constructed. Therefore, the potentials of the form (3) can never occur in the full $\mathcal{N}=4$ supersymmetric gauge theory, though they are possible, in principle, in generic $\mathcal{N}=2$ superconformal theories revealing no extra hidden supersymmetry.

2. The action of $\mathcal{N}=4$ SYM theory can be written in terms of $\mathcal{N}=2$ harmonic superfields as follows

$$S[V^{++},q^+] = \frac{1}{8}\left(\int d^8\zeta_L \operatorname{tr} W^2 + \int d^8\zeta_R \operatorname{tr} \overline{W}^2\right) - \frac{1}{2}\int d\zeta^{(-4)} \operatorname{tr} q^{+a}(D^{++}+igV^{++})q^+_a. \tag{4}$$

The real analytic superfield V^{++} is the harmonic gauge potential of $\mathcal{N}=2$ SYM theory and the analytic superfields q^+_a, $a=1,2$, represent the hypermultiplets (they satisfy the pseudo-reality condition $q^{+a} \equiv \tilde{q}^+_a = \varepsilon^{ab}q^+_b$, where the generalized conjugation $\sim$ was defined in [12]). The $\mathcal{N}=2$ superfield strengths W and $\overline{W}$ are expressed in terms of V^{++}. The superfields V^{++} and q^+_a belong to adjoint representation of the gauge group, g is a coupling constant, $d^8\zeta_L = d^4x\,d^2\theta^+\,d^2\theta^-\,du$, $d^8\zeta_R = d^4x\,d^2\bar\theta^+\,d^2\bar\theta^-\,du$, $d\zeta^{(-4)} = d^4x\,d^2\theta^+\,d^2\bar\theta^-\,du$, du is the measure of integration over the harmonic variables $u^{\pm i}$, $u^{+i}u^-_i = 1$. All other details regarding the action (4), in particular, the precise form of the analyticity-preserving harmonic derivative D^{++}, are given in Refs. [12–14]. We shall basically follow the notation of the book [14].

Either term in (4) is manifestly $\mathcal{N}=2$ supersymmetric. Moreover, the action (4) possesses an extra hidden $\mathcal{N}=2$ supersymmetry which mixes up W, $\overline{W}$ with q^+_a [8,13,14]. As a result, the model under consideration is actually $\mathcal{N}=4$ supersymmetric. Our aim is to examine the possibility of constructing $\mathcal{N}=4$ supersymmetric functionals whose q^+-independent parts would have the form of (1)–(3).

The effective potentials (1)–(3) depend on chiral and anti-chiral abelian strengths W and $\overline{W}$ satisfying the free classical equations of motion $(D^+)^2W = (\overline{D}^+)^2\overline{W} = 0$, where the harmonic projections of the spinor $\mathcal{N}=2$ derivatives $D^i_\alpha, \overline{D}^i_{\dot\alpha}$ are defined as $D^\pm_\alpha = D^i_\alpha u^\pm_i$, $\overline{D}^\pm_{\dot\alpha} = \overline{D}^i_{\dot\alpha}u^\pm_i$. Therefore, in order to construct the above functionals we need to know the hidden $\mathcal{N}=2$ supersymmetry transformations only for on-shell $W, \overline{W}$ and, respectively, for on-shell q^{+a} ($D^{++}q^{+a}=0$). For further use, it is worth to write down the full set of equations for the involved quantities, both on and off shell:
Off-shell:

$$\overline{D}^\pm_{\dot\alpha}W = D^\pm_\alpha\overline{W} = 0, \qquad (D^\pm)^2W = (\overline{D}^\pm)^2\overline{W},$$
$$D^+_\alpha q^{+a} = \overline{D}^+_{\dot\alpha}q^{+a} = 0. \tag{5}$$

On-shell:

$$(D^\pm)^2W = (\overline{D}^\pm)^2\overline{W} = 0,$$
$$D^{++}q^{+a} = D^{--}q^{-a} = 0, \qquad q^{-a} \equiv D^{--}q^{+a},$$
$$D^{++}q^{-a} = q^{+a}, \qquad D^-_\alpha q^{-a} = \overline{D}^-_{\dot\alpha}q^{-a} = 0. \tag{6}$$

In proving the on-shell relations for the hypermultiplet superfield an essential use of the commutation relation $[D^{++}, D^{--}] = D^0$ should be made, with D^0 being the operator which counts harmonic $U(1)$ charges, $D^0 q^{\pm a} = \pm q^{\pm a}$.

From (6) it follows that, in the central basis of the harmonic superspace,

$$q^{\pm a} = q^{ia}(z) u_i^{\pm}, \tag{7}$$

where $q^{ia}(z)$ is the on-shell hypermultiplet superfield independent of harmonic variables and defined on the standard $\mathcal{N} = 2$ superspace with the coordinates $z = (x^m, \theta_{\alpha i}, \bar{\theta}^i_{\dot\alpha})$. Note that in this on-shell description, harmonic variables are to some extent redundant, everything can be formulated in terms of ordinary $N = 2$ superfields $W(z), \overline{W}(z), q^{ia}(z)$. Nevertheless, the use of the harmonic superspace language is still convenient, e.g., because of the opportunity to integrate by parts with respect to the harmonic derivatives in the effective action.

With these remarks taken into account, the on-shell form of the hidden $\mathcal{N} = 2$ transformations reads [14]

$$\begin{aligned}
&\delta W = \frac{1}{2}\bar\epsilon^{\dot\alpha a} \overline{D}^-_{\dot\alpha} q^+_a, \qquad \delta \overline{W} = \frac{1}{2}\epsilon^{\alpha a} D^-_{\alpha} q^+_a, \\
&\delta q^+_a = \frac{1}{4}\big(\epsilon^{\beta}_a D^+_{\beta} W + \bar\epsilon^{\dot\alpha}_a \overline{D}^+_{\dot\alpha} \overline{W}\big), \\
&\delta q^-_a = \frac{1}{4}\big(\epsilon^{\beta}_a D^-_{\beta} W + \bar\epsilon^{\dot\alpha}_a \overline{D}^-_{\dot\alpha} \overline{W}\big),
\end{aligned} \tag{8}$$

where $\epsilon^{\alpha a}, \bar\epsilon^{\dot\alpha a}$ are the Grassmann transformation parameters.

3. We begin with the calculation of $\mathcal{N} = 4$ supersymmetric low-energy effective action corresponding to the non-holomorphic effective potential (1). We assume this action to have the following general form

$$\begin{aligned}
\Gamma\big[W, \overline{W}, q^+\big] &= \int d^{12}z\, du\, \big[\mathcal{H}(W, \overline{W}) \\
&\qquad + \mathcal{L}_q(W, \overline{W}, q^+)\big] \\
&= \int d^{12}z\, du\, \mathcal{L}_{\text{eff}}(W, \overline{W}, q^+).
\end{aligned} \tag{9}$$

Here $d^{12}z$ is the full $\mathcal{N} = 2$ superspace integration measure, $\mathcal{H}(W, \overline{W})$ is given by (1), $\mathcal{L}_q(W, \overline{W}, q^+)$ is some for the moment unknown function which should ensure, together with $\mathcal{H}(W, \overline{W})$, the invariance of the functional (9) with respect to the transformations (8). Notice that the Lagrangian $\mathcal{L}_q(W, \overline{W}, q^+)$, being a function of on-shell superfields, must be in fact independent of the harmonics $u_i^{\pm}$.

The first term in (9) is transformed under (8) as

$$\begin{aligned}
&\delta \int d^{12}z\, du\, \mathcal{H}(W, \overline{W}) \\
&\quad = \frac{1}{2} c \int d^{12}z\, du\, \frac{q^{+a}}{\overline{W} W}\big(\epsilon^{\alpha}_a D^-_{\alpha} W + \bar\epsilon^{\dot\alpha}_a \overline{D}^-_{\dot\alpha} \overline{W}\big).
\end{aligned} \tag{10}$$

Then $\mathcal{L}_q(W, \overline{W}, q^+)$ must be determined from the condition that its variation cancels the variation (10).

We introduce the quantity

$$\mathcal{L}^{(1)}_q \equiv -c \frac{q^{+a} q^-_a}{\overline{W} W} \tag{11}$$

and observe that it transforms according to the rule

$$\begin{aligned}
\delta \frac{q^{+a} q^-_a}{\overline{W} W} &= \frac{q^{+a}}{2\overline{W} W}\big(\epsilon^{\alpha}_{\dot a} D^-_{\alpha} W + \bar\epsilon^{\dot\alpha}_{\dot a} \overline{D}^-_{\dot\alpha} \overline{W}\big) \\
&\quad + \big(q^{+a} q^-_a\big)\delta\left(\frac{1}{\overline{W} W}\right) + D^{--}\left(\frac{\delta q^{+a} q^+_a}{\overline{W} W}\right).
\end{aligned} \tag{12}$$

Let us then consider

$$\begin{aligned}
\mathcal{L}^{(1)}_{\text{eff}} &= \mathcal{H}(W, \overline{W}) - c \frac{q^{+a} q^-_a}{\overline{W} W} \\
&= \mathcal{H}(W, \overline{W}) + \mathcal{L}^{(1)}_q.
\end{aligned} \tag{13}$$

It is easy to see that under the full harmonic $\mathcal{N} = 2$ superspace integral the variation (10) in $\mathcal{L}^{(1)}_{\text{eff}}$ is cancelled by the first term in (12). The variation of (13) generated by the second term in (12) remains non-cancelled. After some algebra, it can be brought into the form

$$\begin{aligned}
&\delta \int d^{12}z\, du\, \mathcal{L}^{(1)}_{\text{eff}} \\
&\quad = \frac{c}{2} \int d^{12}z\, du\, \frac{q^{+b} q^-_b}{(\overline{W} W)^2} \\
&\qquad \times \big(\overline{W} \bar\epsilon^{\dot\alpha a} \overline{D}^-_{\dot\alpha} q^+_a + W \epsilon^{\alpha a} D^-_{\alpha} q^+_a\big) \\
&\quad = -\frac{c}{3} \int d^{12}z\, du\, \frac{q^{+b} q^-_b}{(\overline{W} W)^2} \\
&\qquad \times q^{+a}\big(\bar\epsilon^{\dot\alpha}_a \overline{D}^-_{\dot\alpha} \overline{W} + \epsilon^{\alpha}_a D^-_{\alpha} W\big),
\end{aligned} \tag{14}$$

where we have integrated by parts, used the relations (5), (6) and cyclic identities for the $SU(2)$ doublet indices.

Now let us consider the quantity

$$\mathcal{L}^{(2)}_{\text{eff}} = \mathcal{L}^{(1)}_{\text{eff}} + \frac{c}{3}\left(\frac{q^{+a}q^{-}_{a}}{\overline{W}W}\right)^2 \equiv \mathcal{L}^{(1)}_{\text{eff}} + \mathcal{L}^{(2)}_{q}, \tag{15}$$

where $\mathcal{L}^{(1)}_{\text{eff}}$ is given by (13). The coefficient in the new term $\mathcal{L}^{(2)}_{q}$ has been picked up so that the variation of the numerator of this term cancel (14). The rest of the full variation of $\mathcal{L}^{(2)}_{q}$ once again survives, and in order to cancel it, one is led to add the term

$$\mathcal{L}^{(3)}_{q} = -\frac{2c}{9}\left(\frac{q^{+a}q^{-}_{a}}{\overline{W}W}\right)^3 \tag{16}$$

to $\mathcal{L}^{(1)}_{q} + \mathcal{L}^{(2)}_{q}$, and so on.

The above consideration shows that the q^{+a} dependent part of the full effective action (9), $\mathcal{L}_q = \mathcal{L}_q(W, \overline{W}, q^+)$, should have the form

$$\mathcal{L}_q = \sum_{n=1}^{\infty} \mathcal{L}^{(n)}_{q} = c\sum_{n=1}^{\infty} c_n \left(\frac{q^{+a}q^{-}_{a}}{\overline{W}W}\right)^n \tag{17}$$

with some beforehand unknown coefficients c_n. We have already found $c_1 = -1$, $c_2 = \frac{1}{3}$, $c_3 = -\frac{2}{9}$. The further analysis proceeds by induction.

Let us consider two adjacent terms in the general expansion (17),

$$c_{n-1}\left(\frac{q^{+a}q^{-}_{a}}{\overline{W}W}\right)^{n-1} + c_n\left(\frac{q^{+a}q^{-}_{a}}{\overline{W}W}\right)^{n}, \tag{18}$$

and assume that the variation of the numerator of the first term has been already used to cancel the remaining part of the variation of preceding term (under the integral (9)). Then we prepare the rest of the full variation of the first term like in (14) and demand this part to be cancelled by the variation of the numerator of the second term in (18). This gives rise to the following recursive relation between the coefficients c_{n-1} and c_n:

$$c_n = -2\frac{(n-1)^2}{n(n+1)} c_{n-1} \tag{19}$$

and $c_1 = -1$. This immediately gives

$$c_n = \frac{(-2)^n}{n^2(n+1)}. \tag{20}$$

As the result, the full structure of $\mathcal{L}_q$ is found to be

$$\mathcal{L}_q\big(W, \overline{W}, q^+\big) \equiv \mathcal{L}_q(X) = c\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} X^n$$

$$= c\left\{(X-1)\frac{\ln(1-X)}{X} + \big[\mathrm{Li}_2(X) - 1\big]\right\}, \tag{21}$$

where

$$X = -2\,\frac{q^{+a}q^{-}_{a}}{\overline{W}W} \tag{22}$$

and

$$\mathrm{Li}_2(X) = -\int_0^X \frac{\ln(1-t)}{t}\,dt = \sum_{n=1}^{\infty} \frac{1}{n^2} X^n$$

is Euler dilogarithm [22]. We point out that the expression X (22) does not depend on harmonics u due to the on-shell representation (7)

$$X = -\frac{q^{ia}q_{ia}}{\overline{W}W}. \tag{23}$$

Therefore, $\mathcal{L}_q(X)$ does not depend on harmonics on shell and we can omit the integral over harmonics in the integral (9).

Thus, the full $\mathcal{N}=4$ supersymmetric low-energy effective action for $\mathcal{N}=4$ SYM model with gauge group $SU(2)$ spontaneously broken down to $U(1)$ is given by

$$\Gamma\big[W, \overline{W}, q^+\big] = \int d^{12}z\, \mathcal{L}_{\text{eff}}\big(W, \overline{W}, q^+\big), \tag{24}$$

where

$$\mathcal{L}_{\text{eff}}\big(W, \overline{W}, q^+\big) = \mathcal{H}\big(W, \overline{W}\big) + \mathcal{L}_q(X) \tag{25}$$

with $\mathcal{H}(W, \overline{W})$ and $\mathcal{L}_q(X)$ given, respectively, by (1) and (21), with X (23).[4]

The expression (21) is the exact low-energy result. Indeed, the non-holomorphic effective potential $\mathcal{H}(W, \overline{W})$ (1) is exact, as was argued in [1]. The Lagrangian $\mathcal{L}_q(X)$ (21) was uniquely restored from (1) by $\mathcal{N}=4$ supersymmetry and it is the only one forming, together with $\mathcal{H}(W, \overline{W})$, an invariant of $\mathcal{N}=4$ supersymmetry. Therefore, the functional (24), (25) is the exact low-energy effective action for the theory under consideration.

Now let us turn to the more general non-holomorphic potential (2). Since it is simply a sum of

[4] In principle, the effective action includes the classical action and all quantum corrections. The functional (24) contains only quantum corrections. To write the whole effective action, we have to add the classical action to the functional (24).

the terms analogous to (1), we can repeat the above analysis separately for each term in this sum. As a result, the corresponding $\mathcal{N}=4$ supersymmetric low-energy effective action for $\mathcal{N}=4$ SYM model with gauge group $SU(N)$ spontaneously broken down to $U(1)^{N-1}$ is given by the general expression (24), where $\mathcal{H}(W,\overline{W})$ has the form (2) and

$$\mathcal{L}_{\text{eff}}\big(W,\overline{W},q^+\big)=\sum_{I<J}\mathcal{L}^{IJ}_{\text{eff}}\big(W,\overline{W},q^+\big), \tag{26}$$

with each $\mathcal{L}^{IJ}_{\text{eff}}$ being of the form (21), in which X should be replaced by

$$X_{IJ}=-2\frac{q^{+a}_{IJ}q^{-}_{aIJ}}{W_{IJ}\overline{W}_{IJ}}=-\frac{q^{ia}_{IJ}q_{iaIJ}}{W_{IJ}\overline{W}_{IJ}}. \tag{27}$$

Here

$$W_{IJ}=W^I-W^J,\qquad \overline{W}_{IJ}=\overline{W}^I-\overline{W}^J,$$
$$q^{+a}_{IJ}=q^{+a}_I-q^{+a}_J. \tag{28}$$

The hypermultiplet superfields are $q^{+a}=\sum_I q^{+a}_I e_{II}$, $\sum_I q^{+a}_I=0$, and e_{IJ} is the Weyl basis in the algebra $su(N)$. These hypermultiplet superfields belong to Cartan subalgebra of $su(N)$.

4. It is interesting to elaborate on the component structure of the full effective action (24), (25). We consider only its bosonic part, taking

$$W=\varphi(x)+4i\theta^+_{(\alpha}\theta^-_{\beta)}F^{(\alpha\beta)}(x),$$
$$\overline{W}=\bar\varphi(x)+4i\bar\theta^+_{(\dot\alpha}\bar\theta^-_{\dot\beta)}(x)\overline{F}^{(\dot\alpha\dot\beta)}(x),$$
$$q^{ia}=f^{ia}(x),\qquad D^+_\alpha D^-_\beta W=-4iF_{(\alpha\beta)},$$
$$\overline{D}^+_{\dot\alpha}\overline{D}^-_{\dot\beta}\overline{W}=4i\overline{F}_{(\dot\alpha\dot\beta)}. \tag{29}$$

Here $\varphi(x)$ is the complex scalar field of the $\mathcal{N}=2$ vector multiplet, $F^{\alpha\beta}(x)$ and $\overline{F}^{\dot\alpha\dot\beta}(x)$ are the self-dual and anti-self-dual components of the abelian field strength F_{mn}, and $f^{ia}(x)$ represents four scalar fields of the hypermultiplet $q^{ia}(z)$. In this bosonic approximation, the functional argument X (23) becomes

$$X\big|_{\theta=0}=-\frac{f^{ia}f_{ia}}{|\varphi|^2}\equiv X_0. \tag{30}$$

We also ignore all x-derivatives of the involved fields, since we are interested, as usual, only in the leading part of the expansion of the full effective action in the external momenta.

The component form of the effective action (24) can be then straightforwardly computed by performing integration over θ's. We obtain

$$\int d^{12}z\,\mathcal{L}_{\text{eff}}$$
$$=\frac{1}{16^2}\int d^4x\,(D^+)^2(D^-)^2(\overline{D}^+)^2(\overline{D}^-)^2\mathcal{L}_{\text{eff}}$$
$$\Longrightarrow 4c\int d^4x\,\frac{F^2\overline{F}^2}{|\varphi|^4}\big[1+G(X_0)\big], \tag{31}$$

where

$$G(X_0)=X_0\big[X_0^3\mathcal{L}''''_q(X_0)+8X_0^2\mathcal{L}'''_q(X_0)$$
$$+14X_0\mathcal{L}''_q(X_0)+4\mathcal{L}'_q(X_0)\big]$$
$$=\frac{X_0(2-X_0)}{(1-X_0)^2}, \tag{32}$$

$\mathcal{L}_q(X_0)$ is given by (21) and $F^2=F^{\alpha\beta}F_{\alpha\beta}$, $\overline{F}^2=\overline{F}^{\dot\alpha\dot\beta}\overline{F}_{\dot\alpha\dot\beta}$. The first and second terms in the sum (31) come from $\mathcal{H}(W,\overline{W})$ and $\mathcal{L}_q(X)$, respectively. After substituting the explicit expression (30) for X_0, the bosonic core (31) of the effective action (24) takes the remarkably simple form

$$\Gamma^{\text{bos}}=4c\int d^4x\,\frac{F^2\overline{F}^2}{(|\varphi|^2+f^{ia}f_{ia})^2}. \tag{33}$$

The expression in the denominator is none other than the $SU(4)$ invariant square of 6 scalar fields of $\mathcal{N}=4$ vector multiplet (see, e.g., [23]). After proper obvious redefinitions, it can be cast in the manifestly $SU(4)$-invariant form

$$|\varphi|^2+f^{ia}f_{ia}\sim\phi^{AB}\bar\phi_{AB},\qquad \phi^{AB}=-\phi^{BA},$$
$$\bar\phi_{AB}=\frac12\varepsilon_{ABCD}\phi^{CD},\quad A,B,C,D=1,\ldots,4.$$

This indicates that the full effective action (24), besides being $\mathcal{N}=4$ supersymmetric, possesses also a hidden invariance under the R-symmetry group $SU(4)_{\text{R}}$ of $\mathcal{N}=4$ supersymmetry, quite expected result.[5] In the general case of gauge group $SU(N)$

[5] In (24), only the subgroup $U(2)_{\text{R}}\times SU(2)_{\text{PG}}$ of $\text{SU}(4)_{\text{R}}$ is manifest, with $U(2)_{\text{R}}$ and $SU(2)_{\text{PG}}$ being, respectively, the R-symmetry group of $\mathcal{N}=2$ supersymmetry and the so-called Pauli–Gürsey group [14] acting on the doublet indices of $q^{\pm a}$.

the bosonic effective action is represented by a sum of terms (33), like in (26).

5. Now we wish to inquire whether it is also possible to $\mathcal{N}=4$ supersymmetrize the effective potential (3). The corresponding $\mathcal{N}=4$ supersymmetric effective action must have the following generic form

$$\int d^{12}z\,du\,\{f(V_{IJKL},\overline{V}_{IJKL}) + \mathcal{L}_q(W_{IJ},W_{KL},\overline{W}_{IJ},\overline{W}_{KL},q^{+a}_{IJ},q^{+a}_{KL})\}. \quad (34)$$

Here

$$V_{IJKL}=\frac{W_{IJ}}{W_{KL}},\qquad \overline{V}_{IJKL}=\frac{\overline{W}_{IJ}}{\overline{W}_{KL}}, \quad (35)$$

W_{IJ}, q^{+a}_{IJ} were defined in (28) and $\mathcal{L}_q$ is some unknown function including a dependence on the on-shell hypermultiplet superfields q^{+a}_{IJ}, q^{+a}_{KL}. We are going to show that it is impossible to choose the function $\mathcal{L}_q$ in such a way that the whole functional (34) is invariant under the $\mathcal{N}=4$ supersymmetry transformations (8).

To this end, we start by computing the variation of the first term in (34) under the $\mathcal{N}=4$ transformations (8). This variation can be cast in the form

$$\begin{aligned}\frac{1}{2}\int d^{12}z\,du\,&\frac{\partial^2 f}{\partial V_{IJKL}\partial\overline{V}_{IJKL}}\frac{1}{W_{KL}\overline{W}_{KL}}\\ \times\Big\{&\epsilon^{\alpha a}\left(\frac{W_{IJ}}{W_{KL}}D^-_\alpha W_{KL}-D^-_\alpha W_{IJ}\right)\\ &\times\left(q^+_{aIJ}-\frac{\overline{W}_{IJ}}{\overline{W}_{KL}}q^+_{aKL}\right)\\ &+\bar\epsilon^{\dot\alpha a}\left(\frac{\overline{W}_{IJ}}{\overline{W}_{KL}}\overline{D}^-_{\dot\alpha}\overline{W}_{KL}-\overline{D}^-_{\dot\alpha}\overline{W}_{IJ}\right)\\ &\times\left(q^+_{aIJ}-\frac{W_{IJ}}{W_{KL}}q^+_{aKL}\right)\Big\}\end{aligned} \quad (36)$$

(as in Eq. (3), no summation over I,J,K,L here is assumed). The variation (36) is linear in the hypermultiplet superfields q^{+a}_{IJ}, q^{+a}_{KL}. So, for the variation of $\mathcal{L}_q$ to cancel (36), the function $\mathcal{L}_q$ should start from the term quadratic in hypermultiplet superfields. The most general form of such a term, up to the full harmonic harmonic derivative, is as follows

$$\begin{aligned}\mathcal{L}^{(1)}_q &= g_1\big(q^{+a}_{IJ}q^-_{aIJ}\big)+g_2\big(q^{+a}_{KL}q^-_{aKL}\big)\\ &\quad+g_3\big(q^{+a}_{IJ}q^-_{aKL}\big),\end{aligned} \quad (37)$$

with g_1, g_2, g_3 being some real functions of W_{IJ}, W_{KL}, $\overline{W}_{IJ}$, $\overline{W}_{KL}$. The linear in q^+ part of the full $\mathcal{N}=4$ variation of $\mathcal{L}^{(1)}_q$ (37) is

$$\begin{aligned}\delta\int d^{12}z\,du\,\mathcal{L}^{(1)}_q &= -\frac{1}{2}\int d^{12}z\,du\\ \times\Big\{&\epsilon^{\alpha a}\Big[q^+_{aIJ}\Big(g_1D^-_\alpha W_{IJ}+\frac{1}{2}g_3D^-_\alpha W_{KL}\Big)\\ &+q^+_{aKL}\Big(g_2D^-_\alpha W_{KL}+\frac{1}{2}g_3D^-_\alpha W_{IJ}\Big)\Big]\\ &+\bar\epsilon^{\dot\alpha a}\Big[q^+_{aIJ}\Big(g_1\overline{D}^-_{\dot\alpha}\overline{W}_{IJ}+\frac{1}{2}g_3\overline{D}^-_{\dot\alpha}\overline{W}_{KL}\Big)\\ &+q^+_{aKL}\Big(g_2\overline{D}^-_{\dot\alpha}\overline{W}_{KL}+\frac{1}{2}g_3\overline{D}^-_{\dot\alpha}\overline{W}_{IJ}\Big)\Big]\Big\}.\end{aligned} \quad (38)$$

The functions g_1, g_2, g_3 must be determined from the requirement that the sum of the variations (36) and (38) vanish. However, the direct comparison of these two variations shows that their sum can vanish only provided the additional conditions like

$$\frac{W_{KL}}{W_{IJ}}=\frac{\overline{W}_{KL}}{\overline{W}_{IJ}} \quad (39)$$

hold. They are meaningless, so no appropriate function $\mathcal{L}^{(1)}_q$ exists.

We showed that already in the lowest order in the hypermultiplet superfields it is impossible to find a function $\mathcal{L}_q$, such that its $\mathcal{N}=4$ variation would cancel that of the candidate term (3). Hence, no appropriate $\mathcal{L}^{(q)}$ exists at all. The effective potential of the form (3) cannot appear in $\mathcal{N}=4$ SYM theory, once no its $\mathcal{N}=4$ completion can be defined. One can still expect the appearance of the effective potentials (3) in an arbitrary $\mathcal{N}=2$ superconformal theory.

Thus, the terms (3) are forbidden as contributions to low-energy effective action of $\mathcal{N}=4$ SYM models with gauge group $SU(N)$ spontaneously broken down to $U(1)^{N-1}$. The exact low-energy effective action in the theory under consideration is uniquely specified by the effective Lagrangian (26), (25), (21) obtained by promoting the effective potential (2) to the full $\mathcal{N}=4$ invariant.

6. In summary, in this Letter we addressed the problem of completing the low-energy effective potentials (1)–(3) in $\mathcal{N}=4$ SYM models to the full $\mathcal{N}=4$

supersymmetric invariants. We have shown that such a completion is actually possible only for the potentials (1), (2). The entire effective Lagrangians were found in the explicit form as functions of $\mathcal{N}=2$ superfield strengths and hypermultiplet superfields and they are given by Eqs. (24), (25), (21), (23) and (24), (26), (21), (27), respectively. As for the effective potential (3), we have proved that its promotion to the full $\mathcal{N}=4$ supersymmetric form is impossible. Therefore, the expressions like (3) cannot be regarded as candidate contributions to the effective action of $\mathcal{N}=4$ SYM models. This means, in particular, that just the effective potential (2) and its $\mathcal{N}=4$ completion determine the exact low-energy effective action in the theory under consideration. It is the harmonic superspace approach that made the computations feasible and allowed us to come to these conclusions.

We point out once more that the result (21) was obtained solely on the ground of $\mathcal{N}=4$ supersymmetry as a completion of $\mathcal{N}=2$ supersymmetric effective potential (1) to the full $\mathcal{N}=4$ supersymmetric form. It would be very interesting to reproduce the effective Lagrangian (21) by directly evaluating supergraphs within the quantum $\mathcal{N}=4$ SYM theory. [6]

As the final remark, it is worth noting that the functional argument X (22), (23) has the zero dilatation weight and it is a scalar of the $U(1)$ R-symmetry, since $q^{\pm a}$ and W have the same dilatation weights [14] and $q^{\pm a}$ behave as scalars under the R-symmetry group. So, the full effective action (24) is expected to be invariant under $\mathcal{N}=2$ superconformal symmetry like its pure W, $\overline{W}$ part (1) or (2) [19]. Being also $\mathcal{N}=4$ supersymmetric, this action should respect the whole (on-shell) $\mathcal{N}=4$ superconformal symmetry.

Acknowledgements

The work of the authors was supported in part by INTAS grant No 00-00254. The work of I.L.B. was supported in part by INTAS grant No 991-590. The work of E.A.I. was supported in part by the RFBR grant No 99-02-18417, the RFBR-CNRS grant No 98-02-22034 and NATO Grant PST.CLG 974874.

[6] The supergraphs with the hypermultiplet external lines in $\mathcal{N}=4$ SYM theory at $W=0$ were considered in Ref. [4]. However, according to (21), (22), the full $\mathcal{N}=4$ supersymmetric effective action in both the hypermultiplet and pure $\mathcal{N}=2$ gauge multiplet sectors is ill-defined at $W=0$.

References

[1] M. Dine, N. Seiberg, Phys. Lett. B 409 (1997) 239, hep-th/9705057.

[2] B. de Wit, M.T. Grisaru, M. Roček, Phys. Lett. B 374 (1996) 297, hep-th/9601115;
U. Lindström, F. Gonzalez-Rey, M. Roček, R. von Unge, Phys. Lett. B 388 (1996) 581, hep-th/9607089.

[3] V. Perival, R. von Unge, Phys. Lett. B 430 (1998) 71, hep-th/9801121.

[4] F. Gonzalez-Rey, M. Roček, Phys. Lett. B 434 (1998) 303, hep-th/9804010.

[5] I.L. Buchbinder, S.M. Kuzenko, Mod. Phys. Lett. A 13 (1998) 1629, hep-th/9804168.

[6] I. Chepelev, A.A. Tseytlin, Nucl. Phys. B 511 (1998) 629, hep-th/9705120.

[7] F. Gonzalez-Rey, B. Kulik, I.Y. Park, M. Roček, Nucl. Phys. B 544 (1999) 218, hep-th/9810152.

[8] E.I. Buchbinder, I.L. Buchbinder, S.M. Kuzenko, Phys. Lett. B 446 (1999) 216, hep-th/9810239.

[9] D.A. Lowe, R. von Unge, JHEP 9811 (1998) 014, hep-th/9811017.

[10] M. Dine, J. Gray, Phys. Lett. B 481 (2000) 427, hep-th/9909020.

[11] I.L. Buchbinder, A.Yu. Petrov, Phys. Lett. B 482 (2000) 429, hep-th/0003265.

[12] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky, E. Sokatchev, Class. Quantum Grav. 1 (1984) 469.

[13] A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Class. Quantum Grav. 2 (1985) 601;
A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Class. Quantum Grav. 2 (1985) 613.

[14] A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Harmonic Superspace, Cambridge Univ. Press, 2001.

[15] I.L. Buchbinder, E.I. Buchbinder, E.A. Ivanov, S.M. Kuzenko, B.A. Ovrut, Phys. Lett. B 412 (1997) 309, hep-th/9703147.

[16] E.A. Ivanov, S.V. Ketov, B.M. Zupnik, Nucl. Phys. B 509 (1997) 53, hep-th/9706078.

[17] E.I. Buchbinder, I.L. Buchbinder, E.A. Ivanov, S.M. Kuzenko, Mod. Phys. Lett. A 13 (1998) 1071, hep-th/9803176;
I.L. Buchbinder, I.B. Samsonov, Mod. Phys. Lett. A 14 (1999) 2537, hep-th/9909183;
S. Eremin, E. Ivanov, Mod. Phys. Lett. A 15 (2000) 1859, hep-th/9908054.

[18] I.L. Buchbinder, E.I. Buchbinder, S.M. Kuzenko, B.A. Ovrut, Phys. Lett. B 417 (1998) 61, hep-th/9704214;
I.L. Buchbinder, S.M. Kuzenko, B.A. Ovrut, Phys. Lett. B 433 (1998) 335, hep-th/9710142;
I.L. Buchbinder, S.M. Kuzenko, B.A. Ovrut, Covariant harmonic supergraphity for $N=2$ super-Yang–Mills theories, in: J. Wess, E.A. Ivanov (Eds.), Supersymmetries and Quantum Symmetries, Springer, 1999, pp. 21–36, hep-th/9810040.

[19] I.L. Buchbinder, S.M. Kuzenko, A.A. Tseytlin, Phys. Rev. D 62 (2000) 045001, hep-th/9911221.
[20] S.M. Kuzenko, I.N. McArthur, Phys. Lett. B 506 (2001) 140, hep-th/0101127;
S.M. Kuzenko, I.N. McArthur, Phys. Lett. B 513 (2001) 213, hep-th/0105121.
[21] I.L. Buchbinder, A.Yu. Petrov, A.A. Tseytlin, Two-loop $\mathcal{N}=4$ super-Yang–Mills effective action and interaction between D3-branes, hep-th/0110173.
[22] H. Bateman, A. Erdelyi, Higher Transcendental Functions, Vol. 1, McGraw–Hill, 1953.
[23] M.F. Sohnius, Phys. Rep. 128 (1985) 39.
[24] W. Taylor, Lectures on D-branes, gauge theory and M(atrices), hep-th/9801182;
O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, Phys. Rep. 323 (2000) 183, hep-th/9905111.